

## Problem 2.36

[Difficulty: 4]

**2.36** A flow is described by velocity field  $\vec{V} = at\hat{i} + b\hat{j}$ , where  $a = 0.4 \text{ m/s}^2$  and  $b = 2 \text{ m/s}$ . At  $t = 2 \text{ s}$ , what are the coordinates of the particle that passed through point (2, 1) at  $t = 0$ ? At  $t = 3 \text{ s}$ , what are the coordinates of the particle that passed through point (2, 1) at  $t = 2 \text{ s}$ ? Plot the pathline and streakline through point (2, 1) and compare with the streamlines through the same point at the instants  $t = 0, 1$ , and  $2 \text{ s}$ .

**Given:** Velocity field

**Find:** Coordinates of particle at  $t = 2 \text{ s}$  that was at (2,1) at  $t = 0$ ; coordinates of particle at  $t = 3 \text{ s}$  that was at (2,1) at  $t = 2 \text{ s}$ ; plot pathline and streakline through point (2,1) and compare with streamlines through same point at  $t = 0, 1$  and  $2 \text{ s}$

**Solution:**

**Governing equations:** For pathlines  $u_p = \frac{dx}{dt}$   $v_p = \frac{dy}{dt}$  For streamlines  $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at  $t$ , where  $x_0, y_0$  is the point at which dye is released ( $t_0$  is varied from 0 to  $t$ )

**Assumption:** 2D flow

Given data  $a = 0.4 \frac{\text{m}}{\text{s}^2}$   $b = 2 \frac{\text{m}}{\text{s}^2}$

Hence for pathlines  $u_p = \frac{dx}{dt} = a \cdot t$   $v_p = \frac{dy}{dt} = b$

Hence  $dx = a \cdot t \cdot dt$   $dy = b \cdot dt$

Integrating  $x - x_0 = \frac{a}{2} \cdot (t^2 - t_0^2)$   $y - y_0 = b \cdot (t - t_0)$

The pathlines are  $x(t) = x_0 + \frac{a}{2} \cdot (t^2 - t_0^2)$   $y(t) = y_0 + b \cdot (t - t_0)$

These give the position (x,y) at any time  $t$  of a particle that was at  $(x_0, y_0)$  at time  $t_0$

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are  $x(t_0) = x_0 + \frac{a}{2} \cdot (t^2 - t_0^2)$   $y(t_0) = y_0 + b \cdot (t - t_0)$

These gives the streakline at t, where  $x_0, y_0$  is the point at which dye is released ( $t_0$  is varied from 0 to t)

For a particle that was at  $x_0 = 2$  m,  $y_0 = 1$  m at  $t_0 = 0$  s, at time  $t = 2$  s we find the position is (from pathline equations)

$$x = x_0 + \frac{a}{2} \cdot (t^2 - t_0^2) = 2.8 \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 5 \text{ m}$$

For a particle that was at  $x_0 = 2$  m,  $y_0 = 1$  m at  $t_0 = 2$  s, at time  $t = 3$  s we find the position is

$$x = x_0 + \frac{a}{2} \cdot (t^2 - t_0^2) = 3 \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 3 \text{ m}$$

For streamlines  $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot t}$

So, separating variables  $dy = \frac{b}{a \cdot t} \cdot dx$  where we treat t as a constant

Integrating  $y - y_0 = \frac{b}{a \cdot t} \cdot (x - x_0)$  and we have  $x_0 = 2 \text{ m}$   $y_0 = 1 \text{ m}$

The streamlines are then  $y = y_0 + \frac{b}{a \cdot t} \cdot (x - x_0) = \frac{5 \cdot (x - 2)}{t} + 1$

